4.1 Introduction to Matrices

Objectives:
• Students will organize data into matrices
• Students will solve equations using matrices

Sabrina wants to purchase a sports-utility vehicle (SUV). There are many makes and models of SUVs with many price ranges. Sabrina decides to make a list of the different SUVs and organizes them into a matrix.

For presentation purposes I have placed the information into a table. A matrix would look the same.

<table>
<thead>
<tr>
<th></th>
<th>Base Price</th>
<th>Horse Power</th>
<th>Towing Capacity (lbs)</th>
<th>Cargo Space ($ft^3$)</th>
<th>Fuel Economy (mpg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large SUV</td>
<td>$32,450</td>
<td>285</td>
<td>12,000</td>
<td>46</td>
<td>17</td>
</tr>
<tr>
<td>Standard SUV</td>
<td>$29,115</td>
<td>275</td>
<td>8700</td>
<td>16</td>
<td>17.5</td>
</tr>
<tr>
<td>Mid-Size SUV</td>
<td>$27,975</td>
<td>190</td>
<td>5700</td>
<td>34</td>
<td>20</td>
</tr>
<tr>
<td>Compact SUV</td>
<td>$18,180</td>
<td>127</td>
<td>3000</td>
<td>15</td>
<td>26.5</td>
</tr>
</tbody>
</table>
Matrices

4.1 Introduction to Matrices

Objectives:
• Students will organize data into matrices
• Students will solve equations using matrices

A matrix is a rectangular array of variables or constants in horizontal rows or vertical columns, usually enclosed in brackets.

A matrix is usually named with an uppercase letter; however, in computer programming a matrix may be given a name.

Each number or data value has a position and a purpose. Each value in the matrix is called an element.

A matrix is described by its dimensions. A matrix with \( m \) rows and \( n \) columns is a \( m \times n \) matrix.

\[
\begin{bmatrix}
1 & -3 \\
-5 & 18 \\
0 & -2
\end{bmatrix}
\]

This matrix has 3 rows and 2 columns and is read as a \( 3 \times 2 \) matrix.
Matrices

4.1 Introduction to Matrices

Objectives:
• Students will organize data into matrices
• Students will solve equations using matrices

A matrix with only one row is called a **row matrix**.

A matrix with only one column is called a **column matrix**.

A matrix with the same number of rows and columns is called a **square matrix**.

A matrix where every element is zero is called a **zero matrix**. A zero matrix can have any dimension.

**Equal matrices** have the same dimensions and each element is equal to its corresponding element.

\[
\begin{bmatrix}
6 & 0 & 1 \\
3 & 9 & 3 \\
\end{bmatrix} \neq
\begin{bmatrix}
6 & 3 \\
0 & 9 \\
1 & 3 \\
\end{bmatrix}
\]

Why? Different dimensions

\[
\begin{bmatrix}
1 & 2 \\
8 & 5 \\
\end{bmatrix} \neq
\begin{bmatrix}
1 & 8 \\
2 & 5 \\
\end{bmatrix}
\]

Why? Elements not equal

\[
\begin{bmatrix}
5 & 6 & 0 \\
0 & 7 & 2 \\
3 & 1 & 4 \\
\end{bmatrix} =
\begin{bmatrix}
5 & 6 & 0 \\
0 & 7 & 2 \\
3 & 1 & 4 \\
\end{bmatrix}
\]
4.1 Introduction to Matrices

Objectives:
• Students will organize data into matrices
• Students will solve equations using matrices

Solve the following matrix:

\[
\begin{bmatrix}
3x + y \\
2x + 2y + 3z \\
3y + 2z
\end{bmatrix}
= \begin{bmatrix}
4 - z \\
3 \\
5 - x
\end{bmatrix}
\]

OR

\[
\begin{align*}
3x + y + z &= 4 \\
2x + 2y + 3z &= 3 \\
x + 3y + 2z &= 5
\end{align*}
\]

Multiply the first equation by 2: \(6x + 2y + 2z = 8\)

Eliminate y in the first two equations: \(4x - z = 5\)

Multiply the second equation by 3 and the third equation by 2:

\[
\begin{align*}
6x + 6y + 9z &= 9 \\
2x + 6y + 4z &= 10
\end{align*}
\]

Eliminating y in these equations: \(-4x - 5z = 1\)

Eliminating x in these equations: \(z = -1\) and \(x = 1\)  
Substituting: \(y = 2\)

Bookwork: page 156, problems 10-26
Matrices

4.2 Operations with Matrices

Objectives:
• Students will add and subtract matrices
• Students will multiply a matrix by a scalar

Joanne is a hospital dietician and designs menus for her patients to track various nutrients for each diet. Joanne creates a table to track Calories, protein, and fat in a patient’s diet over a three-day period.

<table>
<thead>
<tr>
<th>Day</th>
<th>Breakfast</th>
<th>Lunch</th>
<th>Dinner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Calories</td>
<td>Protein (g)</td>
<td>Fat (g)</td>
</tr>
<tr>
<td>1</td>
<td>566</td>
<td>18</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>482</td>
<td>12</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>530</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

This data can be organized into three matrices, breakfast, lunch, and dinner. The daily totals can then be determined by adding the three matrices.
Matrices

4.2 Operations with Matrices

Objectives:
• Students will add and subtract matrices
• Students will multiply a matrix by a scalar

If matrix A and matrix B are the same dimensions, then matrix A plus matrix B is a matrix with the same dimensions where each element is added to the corresponding element of A and B.

\[
\begin{bmatrix}
4 & -6 \\
2 & 3
\end{bmatrix}
+ 
\begin{bmatrix}
-3 & 7 \\
5 & -9
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 \\
7 & -6
\end{bmatrix}
\]

Matrix B can be subtracted from matrix A.

\[
\begin{bmatrix}
9 & 2 \\
-4 & 7
\end{bmatrix}
- 
\begin{bmatrix}
3 & 6 \\
8 & -2
\end{bmatrix}
= 
\begin{bmatrix}
6 & -4 \\
-12 & 9
\end{bmatrix}
\]
Matrices

4.2 Operations with Matrices

Objectives:
• Students will add and subtract matrices
• Students will multiply a matrix by a scalar

You can multiply any matrix by a scalar. This operation is known as scalar multiplication.

The product of a scalar and a matrix is when each element is multiplied by that scalar.

If \( A = \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}, \) find \( 3A \)

\[
3 \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 24 & -9 \\ 15 & -27 & 6 \end{bmatrix}
\]

Properties of Matrix Operations

Commutative Property of Addition \( A + B = B + A \)

Associative Property of Addition \( (A + B) + C = A + (B + C) \)

Distributive Property \( s(A + B) = sA + sB \)
Matrices

4.2 Operations with Matrices

Objectives:
• Students will add and subtract matrices
• Students will multiply a matrix by a scalar

If $A = \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$, find $5A - 2B$

$$5 \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} - 2 \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 35 & 15 \\ -20 & -5 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 6 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 17 & 3 \\ -26 & -25 \end{bmatrix}$$

Bookwork: page164, problems 14-29
Matrices

Objectives:
• Students will review matrices rules

\[
\begin{bmatrix}
3x + y \\
2x + 2y + 3z \\
3y + 2z
\end{bmatrix} = \begin{bmatrix}
4 - z \\
3 \\
5 - x
\end{bmatrix}
\]

\[
\begin{align*}
3x + y + z &= 4 \\
2x + 2y + 3z &= 3 \\
x + 3y + 2z &= 5
\end{align*}
\]

If \( A = \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} \) and \( B = \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix} \), find \( 5A - 2B \)

\[
5 \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} - 2 \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 35 & 15 \\ -20 & -5 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 6 & 20 \end{bmatrix} = \begin{bmatrix} 17 & 3 \\ -26 & -25 \end{bmatrix}
\]
Matrices

4.3 Multiplying Matrices

Objectives:
• Students will multiply matrices

How do we identify a matrix? By its dimensions.

Different from adding and subtracting matrices, multiplication of matrices can be performed if the columns of the first matrix is equal to the number of rows of the second matrix.

\[ A_{2 \times 5} \cdot B_{5 \times 4} = AB_{2 \times 4} \quad \text{Notice that the column of the first equals the row of the second.} \]

\[ A_{1 \times 3} \cdot B_{4 \times 3} = ? \quad \text{Can not perform multiplication on these matrices.} \]

When we multiply matrices together, we multiply the first row from the first matrix by the first column of the second matrix.

The element \( a_{ij} \) of \( AB \) is the sum of the products of the corresponding elements in row \( i \) of \( A \) and column \( j \) of \( B \).

\[
\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}
\]

????????????????
Matrices

4.3 Multiplying Matrices

Objectives:
• Students will multiply matrices

Find RS is $R = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 6 + (-5) & -18 + (-7) \\ 9 + 20 & -27 + 28 \end{bmatrix}$$

$$\begin{bmatrix} 6 + (-5) & -18 + (-7) \\ 9 + 20 & -27 + 28 \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$$
In a four-team track meet, 5 points were awarded for each first-place finish, 3 points for each second, and 1 point for each third. Find the total number of points for each school.

<table>
<thead>
<tr>
<th>School</th>
<th>First Place</th>
<th>Second Place</th>
<th>Third Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jefferson</td>
<td>8</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>London</td>
<td>6</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>Springfield</td>
<td>5</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Madison</td>
<td>7</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

\[
R = \begin{bmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{bmatrix}
\quad \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}
\]

This is a 4x3 multiplied by a 3x1 = 4x1

\[
RP = \begin{bmatrix} 8 & 4 & 5 \\ 6 & 3 & 7 \\ 5 & 7 & 3 \\ 7 & 5 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 8(5) + 4(3) + 5(1) \\ 6(5) + 3(3) + 7(1) \\ 5(5) + 7(3) + 3(1) \\ 7(5) + 5(3) + 4(1) \end{bmatrix}
\]

\[
= \begin{bmatrix} 57 \\ 46 \\ 49 \\ 54 \end{bmatrix}
\]
Matrices

4.3 Multiplying Matrices

Objectives:
• Students will multiply matrices

Did you notice that when multiplying matrices, the dimensions do not have to be the same? HOWEVER...

When multiplying matrices, be sure to multiply in the order defined.

\[ R \cdot S \neq S \cdot R \]

Use the calculator to solve these matrices...

\[
R = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \text{ and } S = \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} \quad RS = \begin{bmatrix} 6 & 29 \\ 26 & 23 \end{bmatrix} \quad SR = \begin{bmatrix} -11 & 16 \\ 11 & 40 \end{bmatrix}
\]

Properties of Matrix Multiplication

- Associative Property of Matrix Multiplication \((AB)C = A(BC)\)
- Associative Property of Scalar Multiplication \(s(AB) = (sA)B = A(sB)\)
- Left Distributive Property \(C(A+B) = CA+CB\)
- Right Distributive Property \((A+B)C = AC+BC\)

Bookwork: page 172, problems 13-34
Matrices

4.4 Transformation with Matrices

Objectives:
• Students will use matrices to determine coordinates of translated or dilated figures
• Students will use matrix multiplication to find coordinates of reflected or rotated figures

In the old days comic animation was done by drawing a figure over and over and laying each frame on top of the other to create the movement...

As in a GIF file Complex computer graphics literally translates figures to new positions to create movement...
Objectives:

- Students will use matrices to determine coordinates of translated or dilated figures
- Students will use matrix multiplication to find coordinates of reflected or rotated figures

Points on a coordinate plane can be represented by matrices. The ordered pair \((x, y)\) can be placed in a matrix \[
\begin{bmatrix}
x \\ y
\end{bmatrix}.
\]

Polygons can place their vertices into a matrix, called a vertex matrix. A triangle with vertices \(A(3, 2), B(4, -2), C(2, -1)\) can be placed into a vertex matrix \[
\begin{bmatrix}
3 & 4 & 2 \\ 2 & -2 & -1
\end{bmatrix}.
\]

Transformations can be performed from the preimage to the image.

If the preimage and the image are congruent, then the transformation is an isometry.

A translation occurs when the figure is moved from one location to another without changing its size, shape, or orientation. Matrix addition can be used to show a translated figure.
Matrices

4.4 Transformation with Matrices

Objectives:
• Students will use matrices to determine coordinates of translated or dilated figures
• Students will use matrix multiplication to find coordinates of reflected or rotated figures

Find the coordinates of the vertices of the image of quadrilateral QUAD with Q(2, 3), U(5, 2), A(4, -2), and D(1, -1), if it is moved 4 units to the left and 2 units up.

\[
\begin{bmatrix}
2 & 5 & 4 & 1 \\
3 & 2 & -2 & -1
\end{bmatrix}
+ \begin{bmatrix}
-4 & -4 & -4 & -4 \\
2 & 2 & 2 & 2
\end{bmatrix} = \begin{bmatrix}
-2 & 1 & 0 & -3 \\
5 & 4 & 0 & 1
\end{bmatrix} = Q’U’A’D’
\]
Objectives:
- Students will use matrices to determine coordinates of translated or dilated figures
- Students will use matrix multiplication to find coordinates of reflected or rotated figures

\[
\begin{bmatrix}
2 & 5 & 4 & 1 \\
3 & 2 & -2 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & 1 & 0 & -3 \\
5 & 4 & 0 & 1
\end{bmatrix}
\]
Matrices

4.4 Transformation with Matrices

Objectives:
• Students will use matrices to determine coordinates of translated or dilated figures
• Students will use matrix multiplication to find coordinates of reflected or rotated figures

When a figure is enlarged or reduced, the transformation is called a dilatation. A scalar multiplication is used for dilations.

ΔJKL has vertices J(-2, -3), K(-5, 4), and L(3, 2). Dilate ΔJKL so that the perimeter is half the original perimeter. What are the coordinates of ΔJ′K′L′.

\[
\frac{1}{2} \begin{bmatrix}
-2 & -5 & 3 \\
-3 & 4 & 2 \\
\end{bmatrix}
\begin{bmatrix}
-1 & -\frac{5}{2} & 3 \\
-\frac{3}{2} & 2 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
-1 & -\frac{5}{2} & 3 \\
-\frac{3}{2} & 2 & 1 \\
\end{bmatrix}
\]
Matrices

4.4 Transformation with Matrices

Objectives:
• Students will use matrices to determine coordinates of translated or dilated figures
• Students will use matrix multiplication to find coordinates of reflected or rotated figures

A reflection occurs when every point of a figure is mapped to corresponding image points across a line of symmetry. Below are three common reflection matrices.

<table>
<thead>
<tr>
<th>Reflection Matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflection over the:</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Multiply the vertex matrix on the left by</td>
</tr>
</tbody>
</table>

A rotation occurs when a figure moves around a center point, usually the origin.

<table>
<thead>
<tr>
<th>Rotation Matrices</th>
</tr>
</thead>
<tbody>
<tr>
<td>For a counterclockwise rotation:</td>
</tr>
<tr>
<td>Multiply the vertex matrix on the left by</td>
</tr>
</tbody>
</table>

Bookwork: page 179, problems 12-32
Objectives:
• Students will calculate the determinant of a 2 x 2 matrix.
• Students will calculate the determinant of a 3 x 3 matrix.

Note: Please get a grasp on the concepts and rules for determinants. As the unit lesson progresses, further explanation will be forthcoming.

A **determinant** is a value associated to a **square** matrix. A determinant is enclosed between two parallel lines \[\begin{vmatrix} a & b \\ c & d \end{vmatrix} \]. Do not confuse this with absolute value bars.

The determinant of a 2 x 2 matrix is called a **second-order determinant**.

The value of a second-order determinant is found by calculating the difference of the products of the two diagonals.

\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

\[
\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} = (-2)(8) - (5)(6) = -46
\]
Matrices

4.5 Determinants

Objectives:
• Students will calculate the determinant of a 2 x 2 matrix.
• Students will calculate the determinant of a 3 x 3 matrix.

The determinant of a 3 x 3 matrix is called a third-order determinant. The value of a third-order determinant is found by the expansion by minors. The minor of an element is the determinant form3d when the row and column containing that element are deleted.

\[
\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}
\]

\[
\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2 \begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7 \begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3) \begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix} = 21
\]
Matrices

4.5 Determinants

Objectives:
• Students will calculate the determinant of a 2 x 2 matrix.
• Students will calculate the determinant of a 3 x 3 matrix.

Another method to evaluate third order determinants is using diagonals.

Step 1: Re-write the first two columns on the right side of the determinant.

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix} = \begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix}
\]

Step 2: Find the product of the elements from the top row down and to the right. Then find the product of the elements from the bottom row up to the right.

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix} = \begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{vmatrix} = aei + bfg + cdh
\]

Step 3: Find the value of the determinant by adding the products of the first set of diagonals and subtracting the products of the second set diagonals.

\[
aei + bfg + cdh - gec - hfa - idb
\]
Matrices

4.5 Determinants

Objectives:

• Students will calculate the determinant of a 2 x 2 matrix.
• Students will calculate the determinant of a 3 x 3 matrix.

One useful application of determinants is calculating the area of polygons. The area of a triangle with vertices at \((a, b), (c, d),\) and \((e, f)\) is...

\[
\begin{vmatrix}
a & b & 1 \\
c & d & 1 \\
e & f & 1 \\
\end{vmatrix}
\]

Find the area of a triangle with vertices located at \((-1, 6), (2, 4),\) and \((0, 0)\).

\[
A = \frac{1}{2} \begin{vmatrix}
-1 & 6 & 1 \\
2 & 4 & 1 \\
0 & 0 & 1 \\
\end{vmatrix} = \frac{1}{2} \left[ -1 \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix} \right] \\
= \frac{1}{2} \left[ -16 \right] = -8
\]

Since area must be the absolute value; the area of the triangle is 8 square units.

Bookwork: page 186; problems 16-40 evens, emphasis on 40.
Objectives:
• Students will solve a system of two linear equations using Cramer’s Rule.
• Students will solve a system of three linear equations using Cramer’s Rule.

To find the vertex of a triangle, we would have to solve a system of equations with two lines that are the sides of the triangle.

Using substitution or elimination, would require many calculations.

Cramer’s Rule uses determinants to calculate the vertex or intersection of two lines.

\[ \begin{align*}
ax + by &= e \\
 cx + dy &= f
\end{align*} \]

Note that a, b, c, d, e and f represent constants, not variables.

Solve for x using elimination

\[ \begin{align*}
adx + bdy &= e \\
bcx + bdy &= f
\end{align*} \]

Multiply the first equation by d.

Multiply the second equation by b.

\[ \begin{align*}
adx - bcx &= de - bf \\
(ad - bc)x &= de - bf
\end{align*} \]

\[ x = \frac{de - bf}{ad - bc}; \quad ad - bc \neq 0 \]

\[ y = \frac{af - ce}{ad - bc} \]
Matrices

4.6 Cramer’s Rule

Objectives:
• Students will solve a system of two linear equations using Cramer’s Rule.
• Students will solve a system of three linear equations using Cramer’s Rule.

Cramer’s Rule for Two Variables: The solution of the system of linear equations

\[ ax + by = e \]
\[ cx + dy = f \]

Is \((x, y)\) where

\[
x = \frac{|e \ b|}{|f \ a|}, \quad y = \frac{|a \ e|}{|c \ b|}, \quad \text{and} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0
\]

Cramer’s Rule for Three Variables: The solution of the system of linear equations

\[ ax + by + cz = j \]
\[ dx + ey + fz = k \]
\[ gx + hy + iz = l \]

Is \((x, y, z)\) where

\[
x = \frac{|j \ b \ c|}{|k \ e \ f|}, \quad y = \frac{|a \ j \ c|}{|d \ k \ f|}, \quad z = \frac{|a \ b \ j|}{|d \ e \ k|}, \quad \text{and} \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0
\]

Bookwork: page 192; problems 12-30 evens
Internet shopping is on the rise. Companies protect their computers and their users information by using codes. Cryptography is a method where coded messages can only be deciphered by using a “key” to decode the message.

The simplest technique for cryptography is to:

Assign a number to each letter in the alphabet.

Convert your letter message into a numbers matrix and multiply by a coding matrix. Now the message is unreadable unless you have a decoder ring.

To decode the message, multiply the message by the inverse of the coding matrix.
Objectives:
• Students will determine if two matrices are inverses.
• Students will find the inverse of a 2x2 matrix.

Recall that when using real numbers, when we multiply a number by its inverse, the product is the identity, 1.

\[ a \cdot \frac{1}{a} = 1 \]

Also recall that for matrices, \( A \cdot B \neq B \cdot A \); however, if we multiply by an identity matrix, then \( A \cdot I = I \cdot A \).

The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal and 0 for all other elements.

2x2 Identity Matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]

3x3 Identity Matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]
Matrices

4.7 Identity and Inverse Matrices

Objectives:
• Students will determine if two matrices are inverses.
• Students will find the inverse of a 2x2 matrix.

If \( a \cdot \frac{1}{a} = 1 \) and \( A \cdot I = A \) then...

\[
A \cdot A^{-1} = I; \text{ remember from exponents, } x^{-1} = \frac{1}{x}
\]

\[
det[A] \cdot det[A^{-1}] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}; A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } ad - bc \neq 0
\]

Find the inverse of \( C=\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix} \)

\[
|C| = 14 - 15 = -1
\]

\[
c^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} 2 & -5 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix}
\]

\[
\begin{bmatrix} 7 & 5 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} -14 + 15 & 35 - 35 \\ -6 + 6 & 15 - 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Bookwork: page199; problems 10-30even, 39-41, 44
Matrices

4.8 Using Matrices to Solve Systems of Equations

Objectives:
• Students will write matrix equations for systems of equations.
• Students will solve systems of equations using matrix equations.

Given two linear equations, a product of matrices can be written:

\[ 5x + 7y = 11 \]
\[ 3x + 8y = 18 \]

\[ \begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix} \]

Write the matrix on the left to be a product of coefficients and variables.

\[ \begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix} \text{ coefficient matrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} \text{ variable matrix} \quad \begin{bmatrix} 11 \\ 18 \end{bmatrix} \text{ constant matrix} \]

The system of equations is now expressed as a **matrix equation**.

\[ A \cdot X = B \]
Matrices

4.8 Using Matrices to Solve Systems of Equations

Objectives:

• Students will write matrix equations for systems of equations.
• Students will solve systems of equations using matrix equations.

We can now solve this matrix equation.

\[
A \cdot X = B
\]

\[
a \cdot x = b
\]

\[
x = \frac{1}{a} b
\]

\[
X = A^{-1} B
\]

This means the solution is the product of the inverse of the coefficient matrix and the constant matrix.

If a system does not have a solution, what do you expect to see?

If the coefficient matrix equals 0; then, \(A^{-1}\) does not exist. There is no solution.

The inverse of a 3x3 can be tedious to find. Use the \(x^{-1}\) on the calculator to find the inverse.

Also, matrix B must be multiplied on the left by \(A^{-1}\).

Bookwork: page 206; problems 12-30 even, 34